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Bifurcation analysis in Fractional Order of Epidemic Mathematical models

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- **Abstract:**

The current overview focuses on the study of bifurcation analysis in systems of fractional order in epidemic mathematical models. We discuss the standard approach of dynamical system towards bio mathematical science. We discuss various linear and nonlinear epidemic models like Vecter Borne Disease Models, SI Models, SIR Models etc. The aim of this synopsis is to study the different epidemic models which capture the essence of various interactions allowing the outcomes to be more fully understood with the help of Bifurcation Theory. It has very broad perspective to analyze the biomathematics issues using one of the most important branches of mathematics. The main objective of this study is to elaborate Dynamic's tools on Epidemic Models and this type of study will definitely act as bridge between biomedical sciences and mathematical sciences.

The obtained results of this study will be interpreted mathematically and graphically by using Bifurcation Theory.

- **Introduction:**

If the equilibrium points of the mathematical equations appear, vanish, or shift from stability, these qualitative changes will be referred to as bifurcations. A rapid shift in the quantity or type of the system's constant points is termed as bifurcation. At a bifurcation point, bifurcation happens when the type or quantity of equilibrium solutions changes. Fixed points can modify their stability, appear or vanish, or even separate into periodic points. If parameters are altered, the system's qualitative structure may change. In particular, the stability or creation of fixed points is subject to change. The parameter values at which they occur are known as the bifurcation point, and these qualitative changes in the dynamics are referred to as bifurcation

The qualitative structure of the model can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called Bifurcations and the parameter values at which they occur are called Bifurcation Point. Bifurcations are important scientifically they provide models of transitions and instabilities as some control parameter is varied.

1. **Saddle-node bifurcation:** The saddle node bifurcation is the basic mechanism by which fixed points are created or destroyed.

2. **Transcritical Bifurcation:** In Transcritical Bifurcation point may change where a fixed point must exist for all values of parameter and can never be destroyed. Such a fixed point may change its stability as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability.

3. **Pitchfork bifurcation:** This bifurcation is common in physical problem that have symmetry. And its two different types Supercritical pitchfork bifurcation and Subcritical pitchfork bifurcation.

There are many works can be observed on **Hopf Bifurcation** theory and **Neimark Sacker** Bifurcation with fractional operators. Some are listed in the recent research works**.**

Fractional calculus is a framework of the integrals and derivatives of any real or even complex order. It is amplification of the classical and retains many essential features. The fundamental text for the fractional calculus is Leibniz's letter to **L'Hopital** from 1695, which explores the symbolism for differentiation of non-integer order 1/2 . Up to the middle of the 20th century, notable mathematicians such as Laplace (1812), Fourier (1822), Abel (1823–1826), **Liouville** (1832–1837), **Reimann** (1847), and **Grunwald** (1867–1872) made substantial contributions to the theory of fractional calculus. Being generalizations of the classical derivatives and integrals, fractional derivatives and integrals are anticipated to have even more significance. There are many works can be observed on Caputo and Confirmable fractional operators.

- **History and Mathematical Background of Bifurcation theory:**

Bifurcation theory has roots in the early works of Isaac Newton and Gottfried Wilhelm Leibniz, who explored equilibrium and stability in mathematical physics. However, bifurcations as a mathematical concept gained prominence much later. Joseph-Louis Lagrange and William Rowan Hamilton contributed significantly to the mathematical analysis of dynamical systems, laying groundwork for the study of stability. Henri Poincare (1854–1912) made profound contributions by examining

the qualitative behaviour of differential equations. Poincare introduced geometric methods and identified situations where small parameter changes could lead to qualitative changes in system behaviour—an early recognition of bifurcations. In 20^{th} century formalization of bifurcation theory, Andrey Kolmogorov and Nikolay Bogolyubov applied perturbation methods to stability problems in the 1930s and 1940s. Eberhard Hopf (1942) introduced the Hopf bifurcation, where a stable equilibrium loses stability and gives rise to a periodic orbit. In the mid-20th century, bifurcation theory gained more structure through the works of Vladimir Arnold, René Thom (catastrophe theory), and David Ruelle (chaos theory)

- **Recent work:**

1. **A George Maria Selvam, R Dhinesh babu and Ozlem AK Gumus(2019)**

(Stability and Neimark - sacker bifurcation for a discrete system of one - scroll chaotic attractor with fractional order)

The current study examines the dynamical behaviors of a novel fractional order three-dimensional system that has a chaotic attractor of the discretized equivalent of the one-scroll structure. First, the existence and parametric conditions for an analysis of local stability of the model's steady states are discussed. The existence of Neimark-Sacker (NS) bifurcations in the coexistence steady state is then examined using the bifurcation theory, using the time delay as a bifurcation parameter in the discrete fractional order system. Additionally, for biologically significant sets of parameter values in the discretized system, the trajectories, phase diagrams, limit cycles, bifurcation diagrams, attractors for periods 1, 2, 3, 4, 5, and a chaotic attractor with one scroll are displayed. Lastly, a number of numerical examples are provided to confirm the accuracy of the mathematical results, and further rich dynamics of the model are also analyzed.

2. **Figen Kangalgil (2019)**

(Neimark–Sacker bifurcation and stability analysis of a discrete-time prey–predator model with Allee effect in prey)

This study analyzes the Allee effect on the prey population and the dynamical behaviors of a discrete-time prey–predator model. The model's fixed points are studied for existence and topological categorization. It is demonstrated that by selecting an as the bifurcation parameter, the model can experience a Neimark–Sacker bifurcation at the distinct positive fixed point. Using bifurcation theory, the Neimark-Sacker bifurcation's

existence requirements and bifurcation direction are provided. Additionally, a few numerical simulations are provided to bolster the analytical conclusion. The model's phase portraits and bifurcation diagrams are they provided.

3. **Javad Alidousti(2019)**
   (Stability and bifurcation analysis for a fractional prey–predator scavenger model)
   In this study, we explore a fractional prey–predator scavenger model as well as harvesting by a predator and scavenger. It has been proved that the solutions in this system are bounded and positive. When the requirements for the occurrence of a Hopf bifurcation are satisfied, the model experiences a Hopf bifurcation around one of the current equilibria. Findings demonstrate that chaos disappears in this biological model. We get to the conclusion that the stability domain can be expanded under fractional order and that the fractional system is more stable than the classical situation. Furthermore, the chaotic dynamics can be stabilized and controlled by a fractional order derivative and a reasonable amount of prey harvesting. Here we show an expanded numerical simulation that confirms the findings.

4. **M. H. Akrami ·A. Atabaigi (2020)**
   (Hopf and forward bifurcation of an integer and fractional-order SIR epidemic model with logistic growth of the susceptible individuals)
   This study examines the qualitative behavior of a SIR epidemic model with logistic growth of susceptible people at integer and fractional order. First, the integer-order model's solutions' positivity and boundedness are established. It is demonstrated that the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$ in the integer-order model, driven by the basic reproduction number R_0. The integer-order model is shown to demonstrate forward bifurcation and Hopf bifurcation using the techniques of bifurcations theory. The occurrence of forward and Hopf bifurcations in the fractional order model is then examined, along with the development of certain conditions that can ensure the local stability of the model, all in the interest of the stability theory of fractional-order systems. Finally, examples of numerical simulations are provided to bolster the theoretical

5. **Ramesh, P(2020)**
   (Stability and hopf bifurcation analysis of certain fractional order models in mathematical biology)
   The stability analysis of the fractional order prey-predator model with infection, the asymptotic behavior of the fractional order three species prey-predator model, Analysis of the stability of a latency-based fractional order CHIKV infection model Hopf bifurcation, synchronization of a fractional order Butterfly fish chaotic system, and bifurcation analysis of a fractional order VD model have all been investigated.

6. **Attiq Ul Rehman(2022)**
   (Stability Analysis and Bifurcation of Fractional order Modelling of Vector Borne Diseases)
   The Numerical Solution to Mathematical Model of Dengue Transmission with Relapse and Memory was utilized in this exploratory study to analyze the performance of the different models. Additionally, the following kinds of studies have been conducted: Mathematical Analysis of Multi-Compartmental Malaria Transmission Model with Reinfection; Modeling and Analysis of Dengue Infection with Fractional Differential Operators; and Modeling, Analysis, and Numerical Solution to Malaria Fractional Model with Temporary Immunity and Relapse.

7. **Md. Jasim Uddin(2022)**
   (Analysis of Bifurcation and Chaos to Fractional Order Brusselator Model)
   The Brusselator model has been recognized as the Caputo fractional derivative. Firstly, caputo fractional differential equations for the Brusselator model are constructed via a discretization process. We provide a list of topological classifications for the fixed points in this model. We then show analytically that, under certain parametric conditions, a Flip-bifurcation and a Neimark-Sacker (NS) bifurcation are supported by a fractional order Brusselator model. Using central manifold and bifurcation theory, we prove the presence and direction of NS and Flip bifurcations. It has been found that the fractional order

Brusselator model's dynamical behavior is highly sensitive to the initial conditions and parameter values. The effects of fractional-order parameters and step size on the dynamics of the model are examined. Numerical simulations of bifurcations, phase portraits, periodic orbits, invariant closed cycles, quick onset of chaos, and sudden removal of chaos are performed to validate our analytical findings. Lastly, the chaotic course of the system is stopped using a hybrid control strategy.

8. **Chettouh Besma (2024)**
   (Stability and Bifurcations and Control in Fractional order Chaotic System)
   The understanding of confusing behaviors like stability, chaos, and bifurcations has expanded due to the inclusion of fractional-order dynamics in the study of nonlinear systems. It has also created new opportunities in the field of control theory. These systems add a new degree of adaptability and diversity to the modeling of real-world occurrences by involving derivatives and integrals of non-integer order. The stability, bifurcations, and management of chaos in fractional order chaotic systems are the main topics of this thesis. We explained the essential foundational concepts, including fractional derivation, chaos theory, stability of fractional systems, and bifurcation theory, in the first two chapters in order to accomplish our goal.The final two sections of this thesis contain the major results. In these chapters, we provided the necessary and sufficient conditions for stability, demonstrated the existence of Hopf bifurcations in both integer and fractional cases, and demonstrated the impact of fractional order on stability, chaos control, and the critical point location of Hopf's bifurcation points.

- **Scope of Study**

  There is a lot of travel on the path between the biomedical and mathematical disciplines. Interdisciplinary Applied Mathematics is a series that was established as a result of the correspondingly expanded dialogue between the disciplines. This will include high-level texts and monographs written by scholars who are interested in the relationship between dynamical systems and its biological applications.

- **Objectives of proposed research work**

1. To detect the values of parameters at which a system undergoes a significant qualitative change, such as the transition from stability to instability.
2. To analyse the stability of solutions (equilibria or periodic orbits) to determine how they evolve as parameters change.
3. To classify the types of bifurcations (e.g., saddle-node, Hopf, pitchfork) and their implications on the overall dynamics of the system.
4. To provide insights into possible scenarios, such as sudden transitions, oscillations, or chaos, in complex systems.
5. To apply these insights to fields such as physics, engineering, biology, and economics to solve real-world problems involving nonlinear systems.

- **Research Methodology**

The following steps will be including in the methodology.

1. To refer the relevant reference books on dynamical system for bifurcation theory.
2. To refer the various articles/papers which are based on the proposed topic.
3. To develop the biological model using various dynamical tools.
4. To learn the required mathematical software's like MATLAB and using the software's analyse the outcomes.
5. To find useful solution preferably quantitative.
6. To discuss biological interpretation of mathematical results in terms of insights and predictions.
7. To discuss a realistic dynamical representation of the important biological phenomena.
8. To discuss the global existence of the topic.
9. To publish research paper in reputed journals.

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